Writing this out, we find:

$$\frac{\partial z}{\partial y_i} = \frac{\partial f}{\partial x_1}\frac{\partial x_1}{\partial y_i} + \cdots + \frac{\partial f}{\partial x_n}\frac{\partial x_n}{\partial y_i}$$

Let's bring this back down to Earth with an example:

Ex: Let $z = f(x,y) = x^2 + y^2 + xy$ and suppose $x = \sin t$, $y = e^t$. Find $\frac{dz}{dt}$.

Sol: $\nabla f = \langle 2x+y, 2y+x \rangle$

$\vec{G}(t) = \langle \sin t, e^t \rangle = \langle x, y \rangle$

$\nabla f(\vec{G}(t)) = \langle 2\sin t + e^t, 2e^t + \sin t \rangle$

$\frac{d\vec{G}}{dt} = \langle \cos t, e^t \rangle$

$$\frac{dz}{dt} = \nabla f(\vec{G}(t)) \cdot \frac{d\vec{G}}{dt} = 2\sin t\cos t + e^t\cos t + 2e^{2t} + e^t\sin t$$

◇

Ex: Let $z = f(x,y) = x^2 - 2xy + y^2$, $x = r\cos\theta$, $y = r\sin\theta$. Find z_r & z_θ.

Sol: It's sometimes more convenient to leave ∇f in terms of x&y and plug them in at the end.

$\nabla f = \langle 2x-2y, 2y-2x \rangle$, $\vec{G}(r,\theta) = \langle r\cos\theta, r\sin\theta \rangle$.

$\vec{G}_r = \langle \cos\theta, \sin\theta \rangle$, $\vec{G}_\theta = \langle -r\sin\theta, r\cos\theta \rangle$

$$\frac{\partial z}{\partial r} = \nabla f \cdot \vec{G}_r = (2x-2y)\cos\theta + (2y-2x)\sin\theta$$

$$= (2x-2y)(\cos\theta - \sin\theta) = (2r\cos\theta - 2r\sin\theta)(\cos\theta - \sin\theta)$$

$$= 2r(\cos^2\theta - 2\cos\theta\sin\theta + \sin^2\theta) = 2r(1 - \sin 2\theta)$$

$$\frac{\partial z}{\partial \theta} = \nabla f \cdot \vec{G}_\theta = (2x-2y)(-r\sin\theta) + (2y-2x)(r\cos\theta)$$

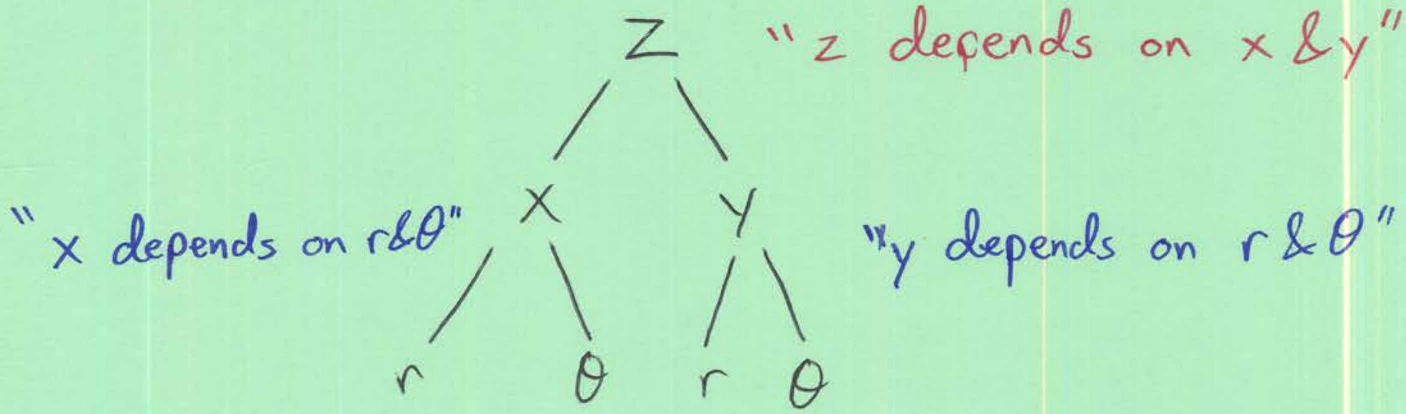
$$= 2r(y-x)(\sin\theta + \cos\theta) = 2r(r\sin\theta - r\cos\theta)(\sin\theta + \cos\theta)$$

$$= 2r^2(\sin\theta - \cos\theta)(\sin\theta + \cos\theta) = 2r^2(\sin^2\theta - \cos^2\theta)$$

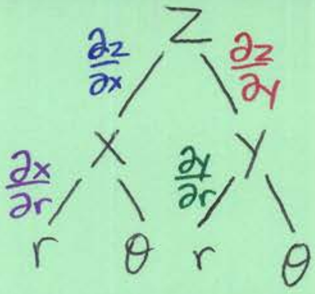
$$= -2r^2\cos 2\theta.$$

$\diamond$

There is a useful bookkeeping method we can use for finding derivatives using the chain rule. This is using dependency trees. As in the last example:



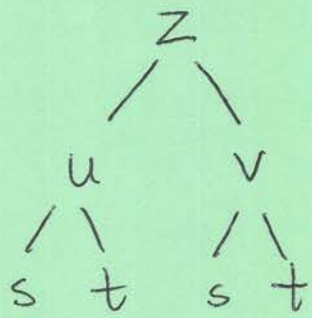
Then, to find, say $\frac{\partial z}{\partial r}$, we follow the paths from z to r, each edge corresponding to a derivative to be taken, then adding up the paths. In this case



$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial r}$$

Ex: Find $\frac{\partial z}{\partial s}$ & $\frac{\partial z}{\partial t}$ where $z = \tan(uv)$, $u = 2s+3t$, and $v = 3s-2t$.

Sol:



So, $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial s} = \left(v\sec^2(uv)\right)(2) + \left(u\sec^2(uv)\right)(3)$

$= \sec^2(uv)\,(2v+3u) = \sec^2\left((2s+3t)(3s-2t)\right)\left(6s-4t+6s+9t\right)$

$= (12s+5t)\sec^2(6s^2+5st-6t^2)$

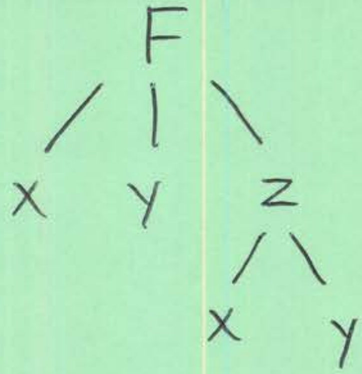
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial t} = \left(v\sec^2(uv)\right)(3) + \left(u\sec^2(uv)\right)(-2)$

$= (3v-2u)\sec^2(uv) = (9s-6t-4s-6t)\sec^2\left((2s+3t)(3s-2t)\right)$

$= (5s-12t)\sec^2(6s^2+5st-6t^2)$ ◇

The chain rule can make implicit differentiation. Let's suppose z is defined implicitly by: $F(x,y,z) = 0$

Using the chain rule:



Taking the derivative with respect to x:

$$\frac{\partial F}{\partial x} + \frac{\partial F}{\partial z}\frac{\partial z}{\partial x} = 0 \Rightarrow \frac{\partial z}{\partial x} = \frac{-F_x}{F_z}$$

Likewise, we can find $\frac{\partial z}{\partial y}$: $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

Ex: Find all first partials of z where z is defined implicitly by: $x^3 z + y\cos z + \frac{\sin y}{z} = 0$.

Sol: $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{3x^2 z}{x^3 - y\sin z - \frac{\sin y}{z^2}} = \frac{-3x^2 z^3}{x^3 z^2 - yz^2\sin z - \sin y}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{\cos z + \frac{\cos y}{z}}{x^3 - y\sin z - \frac{\sin y}{z^2}} = \frac{-(z^2\cos z + z\cos y)}{x^3 z^2 - yz^2\sin z - \sin y}$$

◇

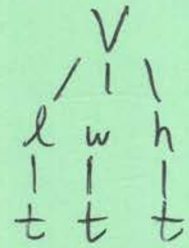
There are technical assumptions one needs for implicit differentiation. These are encapsulated in the implicit function theorem.

<u>Ex</u>: Suppose we have a box containing $42\,\text{in}^3$ of an incompressible fluid. Holding the height of the box fixed at 7in, we squeeze the width of the box so that it decreases at a rate of 1 in/min. How fast is the length of the box changing when the width is 3in?

<u>Sol</u>: Volume: $V = \ell w h$. We want $\frac{d\ell}{dt}$.

$\ell, w, \&\ h$ depend on t. Take $\frac{d}{dt}$ of both sides:

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt}$$



$$= wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt}$$

Fluid incompressible $\Rightarrow \frac{dV}{dt} = 0$, height fixed $\Rightarrow \frac{dh}{dt} = 0$

assumption: $\frac{dw}{dt} = -1$. $h = 7$ & when $w = 3$, $\ell = 2$ since $V = \ell w h = 42$. So:

$$0 = (3)(7)\frac{d\ell}{dt} + (2)(7)(-1) + (2)(3)(0) = 21\frac{d\ell}{dt} - 14$$

$$\Rightarrow \frac{d\ell}{dt} = \frac{14}{21}\ \text{in/min} = \frac{2}{3}\ \text{in/min}.$$

◇

14.6 - Directional Derivatives

The partial derivatives give slopes in the x- & y- direction but what about all the other directions? Pick a direction by choosing a unit vector $\vec{u} = \langle a, b \rangle$, then the directional derivative of $f = f(x,y)$ in the direction $\vec{u}$ is

$$\boxed{D_{\vec{u}} f(x,y) = \lim_{h \to 0} \frac{f(x+ah, y+bh) - f(x,y)}{h} = \nabla f(x,y) \cdot \vec{u}}$$

Writing $\vec{x} = \langle x, y \rangle$, we can write $D_{\vec{u}} f(\vec{x}) = \lim_{h \to 0} \frac{f(\vec{x} + h\vec{u}) - f(\vec{x})}{h}$.
Naturally, this extends to functions of more variables.
Again, $\vec{u}$ must be a unit vector!

Ex: Find the directional derivative of $f(x,y) = e^x \sin y$ at $(3, \frac{\pi}{3})$ in the direction of $\vec{v} = \langle 1, -2 \rangle$.

Sol: $\nabla f = \langle e^x \sin y, e^x \cos y \rangle \Rightarrow \nabla f(3, \frac{\pi}{3}) = \langle e^3 \cdot \frac{1}{2}, e^3 \cdot \frac{\sqrt{3}}{2} \rangle = \frac{e^3}{2} \langle 1, \sqrt{3} \rangle$

$\vec{v}$ is not a unit vector, so we have to make it one:

$$\hat{v} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{\langle 1, -2 \rangle}{\sqrt{1^2 + (-2)^2}} = \frac{1}{\sqrt{5}} \langle 1, -2 \rangle$$

$$D_{\hat{v}} f(3, \tfrac{\pi}{3}) = \nabla f(3, \tfrac{\pi}{3}) \cdot \hat{v} = \frac{e^3}{2\sqrt{5}} \left[(1)(1) + (\sqrt{3})(-2) \right] = \frac{e^3}{2\sqrt{5}} (1 - 2\sqrt{3})$$

◇

We can ask: in what direction is f changing the fastest? Let $\vec{u}$ be a unit vector & θ the angle between ∇f & $\vec{u}$.

$\|\vec{u}\|=1$

Then, $D_{\vec{u}}f = \nabla f \cdot \vec{u} = \|\nabla f\| \|\vec{u}\| \cos\theta = \|\nabla f\| \cos\theta$

This is maximized when $\theta = 0$ ($\cos\theta = 1$), meaning that $\vec{u}$ points in the same direction as ∇f. Moreover, we see that the maximum rate of change is $\|\nabla f\|$.

Ex: Find the maximum rate of change of $f(x,y,z) = \dfrac{x+y}{z}$ at $(1,1,-1)$, and the direction in which it occurs.

Sol: $\nabla f = \left\langle \dfrac{1}{z}, \dfrac{1}{z}, \dfrac{-(x+y)}{z^2} \right\rangle$, $\nabla f(1,1,-1) = \langle -1,-1,-2 \rangle$

The maximum rate of change is:

$\|\nabla f(1,1,-1)\| = \|\langle -1,-1,-2\rangle\| = \sqrt{1+1+4} = \sqrt{6}$

And the direction is:

$$\vec{u} = \frac{\nabla f(1,1,-1)}{\|\nabla f(1,1,-1)\|} = \frac{-1}{\sqrt{6}}\langle 1,1,2\rangle.$$

$\Diamond$

Geometric Uses of the Gradient

Suppose we have a surface S given as a level surface of some function $F(x,y,z)$. So S is the graph of $F(x,y,z)=k$. Let's take a point (x_0,y_0,z_0) on S. To get a tangent vector to S at (x_0,y_0,z_0), take a curve $\vec{r}(t)$ on S passing through the point (i.e., $F(\vec{r}(t))=k$ & $\vec{r}(t_0)=\langle x_0,y_0,z_0\rangle$), then take its derivative at that point. That is, the tangent vector is $\vec{r}\,'(t_0)$. If $\vec{r}(t) = \langle x(t), y(t), z(t) \rangle$, then by taking $\frac{d}{dt}$ of $F(\vec{r}(t)) = k$ gives

$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} = 0$$

components of ∇

components of $\vec{r}\,'(t)$

$\Rightarrow \nabla F \cdot \vec{r}\,'(t) = 0$, i.e., ∇F is perpendicular to every tangent vector on S, i.e., ∇F is perpendicular to S.

This, of course, applies to show ∇G is perpendicular to level curves of $G(x,y)$.

Lecture 10

Ex: Estimate the gradient at the indicated points given a contor plot of $f = f(x,y)$. (See next page.)

Now, since ∇F is perpendicular to level surfaces of F, we can use it to find tangent planes and normal lines to the surface.